MTH 112 **Test 5**
Fall 2014

Calculus II

Name ___________________________

Justify all answers by showing your work or by providing a coherent explanation. and please circle your answers

1. Find the sum $\sum_{k=1}^{\infty}\frac{3}{5^k}$ $= 3\sum_{k=1}^{\infty}\left(\frac{1}{5}\right)^k = 3\left[\sum_{k=0}^{\infty}\left(\frac{1}{5}\right)^k - 1\right] = 3\left[\sum_{k=0}^{\infty}\left(\frac{1}{5}\right)^k - 1\right] =$

$$3\left[\frac{1}{1-\frac{1}{5}} - 1\right] = \frac{3}{4}$$

2. Find the first four non-zero terms of the Maclaurin expansion of $f(x) = \sinh x$. Also, show that the answer for the former is equivalent to the series expansion for $\frac{1}{2}e^x - \frac{1}{2}e^{-x}$.

$f(x) = \sinh x$ $\quad f(0) = 0$
$f'(x) = \cosh x$ $\quad f'(0) = 1$
$f''(x) = \sinh x$ $\quad f''(0) = 0$
$f'''(x) = \cosh x$ $\quad f'''(0) = \cosh x$
$f^{(4)}(x) = \sinh x$ $\quad f^{(4)}(0) = 0$
$f^{(5)}(x) = \cosh x$ $\quad f^{(5)}(0) = 1$
$f^{(6)}(x) = \sinh x$ $\quad f^{(6)}(0) = 0$
$f^{(7)}(x) = \cosh x$ $\quad f^{(7)}(0) = 1$
$f^{(8)}(x) = \sinh x$ $\quad f^{(8)}(0) = 0$

So $f(x) = \sinh x \approx x + \frac{x^3}{3!} + \frac{x^5}{5!} + \frac{x^7}{7!}$

$$\frac{1}{2}e^x = \frac{1}{2} + \frac{1}{2}x + \frac{1}{2}\left(\frac{x^2}{2!}\right) + \frac{1}{2}\left(\frac{x^3}{3!}\right) + \frac{1}{2}\left(\frac{x^4}{4!}\right) + \frac{1}{2}\left(\frac{x^5}{5!}\right) + \frac{1}{2}\left(\frac{x^6}{6!}\right)\cdots$$

$$\frac{1}{2}e^{-x} = \frac{1}{2} + \frac{1}{2}(-x) + \frac{1}{2}\left(\frac{(-x)^2}{2!}\right) + \frac{1}{2}\left(\frac{(-x)^3}{3!}\right) + \frac{1}{2}\left(\frac{(-x)^4}{4!}\right) + \frac{1}{2}\left(\frac{(-x)^5}{5!}\right) + \frac{1}{2}\left(\frac{(-x)^6}{6!}\right)\cdots$$

$$= \frac{1}{2} - \frac{1}{2}x + \frac{1}{2}\left(\frac{x^2}{2!}\right) - \frac{1}{2}\left(\frac{x^3}{3!}\right) + \frac{1}{2}\left(\frac{x^4}{4!}\right) - \frac{1}{2}\left(\frac{x^5}{5!}\right) + \frac{1}{2}\left(\frac{x^6}{6!}\right)\cdots$$

$$\therefore\ \frac{1}{2}e^x - \frac{1}{2}e^{-x} \approx\approx x + \frac{x^3}{3!} + \frac{x^5}{5!} + \frac{x^7}{7!}$$

3. Find the first four non-zero terms of the Maclaurin expansion of $f(x) = (2 - x)^3$.

$f(x) = (2-x)^3 \qquad f(0) = 8$

$f'(x) = -3(2-x)^2 \qquad f'(0) = -12$

$f''(x) = 2 \cdot 3(2-x) \qquad f''(0) = 12$

$f'''(x) = -2 \cdot 3 \qquad f'''(0) = -6$

So $f(x) = (2-x)^3 = 8 - 12x + 6x^2 - x^3$

4. Find the interval of convergence for $\sum_{k=1}^{\infty} (-1)^{k+1} \frac{x^k}{k(2^k)}$.

$\lim_{k\to\infty} \left| \frac{x^{k+1}}{(k+1)(2^{k+1})} \cdot \frac{(k)(2^k)}{x^k} \right| = \lim_{k\to\infty} \left| \frac{k}{2(k+1)} \cdot x \right| = \frac{1}{2}|x|$. So $\frac{1}{2}|x| < 1$ when $|x| < 2$. Now check endpoints -

1) $x = 2$: $\sum_{k=1}^{\infty} (-1)^{k+1} \frac{2^k}{k(2^k)} = \sum_{k=1}^{\infty} (-1)^{k+1} \frac{1}{k}$ which converges by the alternating series test.

2) $x = -2$: $\sum_{k=1}^{\infty} (-1)^{k+1} \frac{(-2)^k}{k(2^k)} = \sum_{k=1}^{\infty} -\left(\frac{1}{k}\right)$ which diverges. Therefore the interval of convergence is $-2 < x \le 2$

5. Find the interval of convergence for $1 - x + \frac{x^2}{2^2} - \frac{x^3}{3^2} + \frac{x^4}{4^2} - \cdots$. This series is $\sum_{k=0}^{\infty} (-1)^k \frac{x^k}{k^2}$.

Using the ratio test we have $\lim_{k \to \infty} \left| \frac{x^{k+1}}{(k+1)^2} \cdot \frac{k^2}{x^k} \right| = \lim_{k \to \infty} \left| \frac{k^2}{(k+1)^2} \cdot x \right| = |x|$. So $|x| < 1$.

Now check endpoints -

1) $x = 1$: $\sum_{k=1}^{\infty} (-1)^k \frac{1^k}{k^2} = \sum_{k=1}^{\infty} (-1)^k \frac{1}{k^2}$ which converges by the alternating series test.

1) $x = -1$: $\sum_{k=1}^{\infty} (-1)^k \frac{(-1)^k}{k^2} = \sum_{k=1}^{\infty} \left(\frac{1}{k}\right)$ which diverges. Therefore the interval of

convergence is $-1 < x \leq 1$

6. Use power series to represent the first four (4) terms of $\int \frac{\sin x}{x} dx$.

$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \frac{x^9}{9!} \cdots$, $\frac{\sin x}{x} = 1 - \frac{x^2}{3!} + \frac{x^4}{5!} - \frac{x^6}{7!} + \frac{x^6}{9!} \cdots$

So, $\int \frac{\sin x}{x} = \int 1 - \frac{x^2}{3!} + \frac{x^4}{5!} - \frac{x^6}{7!} + \frac{x^6}{9!} \cdots = x - \frac{1}{3} \cdot \frac{x^3}{3!} + \frac{1}{5} \cdot \frac{x^5}{5!} - \frac{1}{7} \cdot \frac{x^7}{7!}$

7. The ratio test for convergence of numerical series succeeds if the result is less than 1. If the result is greater than 1, then the series will diverge. What if the result is equal to one? Try the ratio test on two series you know will converge and diverge respectively. What should then be said if the result of the ration test is equal to one?

The ratio test applied to diverget series $\sum_{k=1}^{\infty} \frac{1}{k}$ yields 1 and so does the ratio test applied to the convergent series $\sum_{k=1}^{\infty} \frac{1}{k^2}$. So the ratio test is **inconclusive** if it yields a value equal to 1 !

8. For each of the following tell whether the series converges or diverges and **explain why** you have answered accordingly.

a) $\sum_{k=0}^{\infty} \frac{2k^2 - 3k + 1}{5k^7 - 18k^5 + k + 12} \sim \sum_{k=0}^{\infty} \frac{1}{k^5}$ therefore it **converges**

c) $\sum_{k=0}^{\infty} \frac{1}{3^k - 2k}$ is dominated by convergent series $\sum_{k=0}^{\infty} \frac{4}{3^k}$, therefore it **converges**

b) $\sum_{k=0}^{\infty} \sin\left(\frac{\pi}{2}k\right)$

$\left\{\sin\left(\frac{\pi}{2}k\right)\right\}_{k=0}^{\infty}$ does not converge to zero, therefore this series **diverges**

d) $\sum_{k=3}^{\infty} \frac{(-1)^k}{k-2}$ passes the alternating series test, therefore it **converges**

9. (4 *points*) What is the equation that Leonhard Euler stated in his proof that God exists?

$e^{\pi i} + 1 = 0$